

CFT on the Brane with a Reissner-Nordstrom-de Sitter Twist

by

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ABSTRACT

We consider a brane universe in a Reissner-Nordstrom-de Sitter background spacetime of arbitrary dimensionality. It is shown that the brane evolution is described by generalized Friedmann equations for radiative matter along with an electrostatic potential. On the basis of the (conjectured) dS/CFT correspondence, we identify various thermodynamic properties of the brane. It is then demonstrated that, when the brane crosses the de Sitter cosmological horizon, the CFT thermodynamics and Friedmann-like equations coincide. Moreover, the CFT entropy is shown to be expressible in a generalized Cardy-Verlinde form. Finally, we consider the holographic entropy bounds in this scenario.

1 Introduction

The so-called “holographic principle” [1, 2] has had considerable influence on recent and current gravitational physics. The premise of this principle, roughly speaking, is that the maximal entropy in any given volume will be determined by the largest black hole that fits inside of that volume. On this basis, it has been argued that all relevant degrees of freedom of any given system must, in some sense, “live” on the boundary of the system.

One of the more prolific manifestations of the holographic principle has been the AdS/CFT correspondence [3, 4, 5]. More specifically, it has been convincingly argued that the thermodynamics at the horizon of a $n+2$ -dimensional anti de-Sitter (AdS) black hole can be identified with a certain $n+1$ -dimensional conformal field theory (CFT). This dual CFT is assumed to be a strongly-coupled one and to live on a timelike surface that can be identified as an asymptotic boundary of the AdS spacetime.

In a breakthrough paper [6], Verlinde directly applied the above ideas to an $n+1$ -dimensional radiation-dominated, closed Friedmann-Robertson-Walker (FRW) universe.¹ This paper covered much ground, but the following two discoveries are of particular interest. (i) The AdS/CFT correspondence leads to an entropy that can be expressed in terms of a generalized Cardy formula [19]; with the Cardy central charge being directly related to the Casimir energy.² (ii) When the “Casimir entropy” saturates a certain bound (identifiable with the holographic Bekenstein-Hawking entropy [20, 21] of a universal-size black hole), then the Friedmann equations coincide precisely with the generalized Cardy formula. To express this more elegantly, the CFT and FRW equations remarkably merge at a holographic saturation point; thus implying that both sets of equations arise from some fundamental, underlying theory.

In Ref.[22], Savonije and Verlinde have extended these notions to the case of a Randall-Sundrum brane world [23, 24] in the background of an AdS-Schwarzschild (black hole) geometry. In this scenario, the $n+1$ -dimensional CFT is regarded as living on the brane, which serves as a boundary for the $n+2$ -dimensional AdS bulk spacetime. Given a suitable choice of boundary

¹For earlier studies on holography in a cosmological context, see Refs.[7]- [18].

²In this context, the Casimir energy refers specifically to the sub-extensive contribution to the thermal energy. Furthermore, we will refer to the corresponding entropy as the “Casimir entropy”.

conditions, Savonije and Verlinde have shown that the brane world corresponds to a FRW universe, with the brane dynamics being described by the Friedmann equations for radiative matter. Moreover, they demonstrated that the CFT thermodynamic relations coincide with the Friedmann equations at the point when the brane crosses the black hole (outermost) horizon.

Let us note that the Verlinde-Savonije treatment has seen been extended and generalized in a large array of studies [25]-[43].

In analogy to the highly successful AdS/CFT duality, a dS/CFT correspondence has also been hypothesized [44].³ Although de Sitter (dS) space is related to AdS via a simple sign change in the cosmological constant, there are dire implications that make the proposed dS/CFT duality a significantly more challenging prospect. For instance, dS space lacks a globally time-like Killing vector and the black hole horizon (and related properties) have an ambiguous observer dependence. (For a comprehensive discussion on dS spacetimes, see Ref.[49].)

In spite of the inherent geometrical complications, there has still been substantial progress towards a holographic understanding of dS space [44]-[69]. With respect to the proposed dS/CFT correspondence, the higher-dimensional horizon is taken to be the dS cosmological horizon, while the dual CFT is regarded as a Euclidean one that lives on a spacelike asymptotic boundary.

Very recently, work has begun on adapting the Verlinde-Savonije program [6, 22] to a dS holographic picture [67, 68, 69]. Given that recent observational evidence points to us living in a dS universe [70], the importance of such studies probably cannot be understated. On this note, the purpose of the current paper is to see if the outcomes of Ref.[22] hold up for a yet-to-be-considered scenario. Namely, a brane universe in a Reissner-Nordstrom-de Sitter (RNdS) background.⁴

The rest of the paper proceeds as follows. In Section 2, we identify the thermodynamics properties of the RNdS cosmological horizon. This is followed by a formulation of the brane dynamics, which leads to Friedmann-like equations. In Section 3, we apply the dS/CFT correspondence to obtain the thermodynamics of the Euclidean CFT that lives on the brane. Also, the

³For earlier works in this regard, see Refs.[45]-[48].

⁴By incorporating charge into the model, we are following the work of Cai et al. [37] and (in particular) Youm [41]; both of whom considered a charged AdS background.

Friedmann equations are rewritten in a form for which their connection with CFT thermodynamics is manifest. In Section 4, we consider when the brane crosses the cosmological horizon and consequently demonstrate that the CFT thermodynamics and Friedmann equations coincide at precisely this point. Furthermore, the CFT entropy is shown to be expressible in a generalized Cardy-Verlinde [19, 6] form. Section 5 considers the implications of the prior analysis with regard to holographic entropy bounds. Finally, Section 6 closes with a summary and brief discussion.

2 Bulk Thermodynamics and Brane Cosmology

Let us begin by considering an $n+1$ -dimensional brane of constant tension in an $n+2$ -dimensional Reissner-Nordstrom-de Sitter (RNdS) background. In a suitably static gauge, the bulk solution can be written as follows [71]:

$$ds_{n+2}^2 = -h(a)dt^2 + \frac{1}{h(a)}da^2 + a^2 d\Omega_n^2, \quad (1)$$

$$h(a) = 1 - \frac{a^2}{L^2} - \frac{\omega_{n+1}M}{a^{n-1}} + \frac{n\omega_{n+1}^2 Q^2}{8(n-1)a^{2n-2}}, \quad (2)$$

$$\omega_{n+1} = \frac{16\pi G_{n+2}}{nV_n}, \quad (3)$$

$$\phi_{dS}(a) = \frac{n}{4(n-1)} \frac{\omega_{n+1}Q}{a^{n-1}}. \quad (4)$$

Here, L is the curvature radius of the dS background, $d\Omega_n^2$ is a unit n -sphere with volume V_n , G_{n+2} is the $n+2$ -dimensional Newton constant, M and Q represent the conserved quantities of black hole mass and charge, and $\phi_{dS}(a)$ is a measure of the electrostatic potential at a .⁵

In any dS space, there is a well-defined cosmological horizon having similar thermodynamic properties to that of a black hole horizon [49]. In our

⁵Since there is no spatial infinity in RNdS space to use as a reference point, ϕ_{dS} should not be interpreted as a literal electrostatic potential, as it was in analogous AdS studies [37, 41].

case, this cosmological horizon ($a = a_H$) corresponds to the largest root of $h(a) = 0$. Thus, the following useful relation can be obtained:

$$1 - \frac{a_H^2}{L^2} - \frac{\omega_{n+1}M}{a_H^{n-1}} + \frac{n\omega_{n+1}^2Q^2}{8(n-1)a_H^{2n-2}} = 0. \quad (5)$$

In analogy with black hole thermodynamics [72], the cosmological horizon has an associated temperature, entropy and chemical potential that are respectively given as follows:⁶

$$T_{dS} = \frac{(n+1)a_H^2 - (n-1)L^2}{4\pi L^2 a_H} + \frac{n\omega_{n+1}^2 Q^2}{32\pi a_H^{2n-1}}, \quad (6)$$

$$S_{dS} = \frac{a_H^n V_n}{4G_{n+2}}, \quad (7)$$

$$\Phi_{dS} = \phi_{dS}(a_H) = \frac{n}{4(n-1)} \frac{\omega_{n+1}Q}{a_H^{n-1}}. \quad (8)$$

The premise of the dS/CFT correspondence is that the above thermodynamics can be identified, in some appropriate manner, with a Euclidean CFT that resides (in the RNdS space) on a spacelike boundary at temporal infinity. We will have more to say on this later.

Let us now consider the brane, which can be regarded as a dynamical boundary of the RNdS geometry. The brane dynamics are to be described here via the following boundary action:

$$\mathcal{I}_b = \frac{1}{8\pi G_{n+2}} \int_{\partial\mathcal{M}} \sqrt{-g^{ind}} \mathcal{K} + \frac{\sigma}{8\pi G_{n+2}} \int_{\partial\mathcal{M}} \sqrt{-g^{ind}}, \quad (9)$$

where g_{ij}^{ind} is the induced metric on the boundary ($\partial\mathcal{M}$), $\mathcal{K} \equiv \mathcal{K}_i^i$ is the trace of the extrinsic curvature and σ is a parameter measuring the brane tension. Varying this action with respect to the induced metric, one obtains the following equation of motion:

$$\mathcal{K}_{ij} = \frac{\sigma}{n} g_{ij}^{ind}. \quad (10)$$

⁶In particular, the inverse temperature can be identified with the periodicity of Euclidean time and the entropy with one quarter of the horizon surface area [72]. The chemical potential has been defined in analogy to the RN-AdS case [41].

Following Ref.[22], we can clarify the brane dynamics by introducing a new (Euclidean) time parameter, τ ; whereby $a = a(\tau)$, $t = t(\tau)$ and:

$$\frac{1}{h(a)} \left(\frac{da}{d\tau} \right)^2 - h(a) \left(\frac{dt}{d\tau} \right)^2 = 1. \quad (11)$$

Unlike in the original analysis [22], we have chosen τ so that the resulting line element is spacelike. This choice naturally reflects the dS duality with a Euclidean CFT [44].

Substituting the above condition into Eq.(1), we see that the induced brane metric takes on a Euclidean FRW form. That is:

$$ds_{n+1}^2 = d\tau^2 + a^2(\tau) d\Omega_n^2. \quad (12)$$

With this formulation, it is clear that the radial distance, $a = a(\tau)$, measures the size of the $n+1$ -dimensional brane universe.

Let us now return considerations to the equation of motion (10). The extrinsic curvature can readily be calculated (see, for instance, Ref.[73]) and then expressed in terms of the functions $a(\tau)$ and $t(\tau)$. This process yields:

$$\left(\frac{dt}{d\tau} \right)^2 = \frac{\sigma^2 a^2}{h^2(a)}. \quad (13)$$

Next, we define the Hubble parameter in the usual way: $H \equiv \dot{a}/a$.⁷ Then Eq.(11) can be re-expressed as follows:

$$H^2 = \frac{1}{a^2} - \frac{1}{L^2} - \frac{\omega_{n+1} M}{a^{n+1}} + \frac{n\omega_{n+1}^2 Q^2}{8(n-1)a^{2n}} + \sigma^2, \quad (14)$$

where Eqs.(2,13) have also been incorporated. We are free to set $\sigma^2 = 1/L^2$ and conveniently cancel off the a -independent terms. This choice yields a generalized (first) Friedmann equation:

$$H^2 = \frac{1}{a^2} - \frac{\omega_{n+1} M}{a^{n+1}} + \frac{n\omega_{n+1}^2 Q^2}{8(n-1)a^{2n}}. \quad (15)$$

⁷Dots will always denote differentiation with respect to τ .

Furthermore, we can take the τ derivative of the above equation to obtain the corresponding second Friedmann equation:

$$\dot{H} = -\frac{1}{a^2} + \frac{(n+1)\omega_{n+1}M}{2a^{n+1}} - \frac{n^2\omega_{n+1}^2Q^2}{8(n-1)a^{2n}}. \quad (16)$$

Note that the bulk RNdS background effectively induces both radiative matter ($\sim M/a^{n+1}$) and stiff matter ($\sim Q^2/a^{2n}$) in the brane universe [31].

3 Euclidean CFT on the Brane

Let us re-establish the premise of the dS/CFT correspondence as it applies to our model. The thermodynamic properties of the cosmological horizon can presumably be identified with the thermodynamics of a dual CFT that is Euclidean and living on the brane. In analogy to the AdS analysis of Savonije and Verlinde [22], we will use this proposed duality in defining the brane (CFT) thermodynamics. Although the process is not so well defined for dS spacetimes, we can proceed (at least naively) by making some conjectural identifications.

We begin here by applying the knowledge that the metric for a boundary CFT can only be determined up to a conformal factor [4, 5]. Following Ref.[22], we first consider the asymptotic form of the Euclidean RNdS metric:

$$\lim_{a \rightarrow \infty} \left[\frac{L^2}{a^2} ds_{n+2}^2 \right] = dt_E^2 + L^2 d\Omega_n^2. \quad (17)$$

Evidently, the Euclidean CFT time is only equal to the Euclidean RNdS time (t_E) when the radius of the spatial sphere has been set equal to L . If we want the sphere to have a radius equal to a , then the Euclidean CFT time should be set equal to at_E/L . Proceeding on this basis, it follows that the same factor a/L will appear in the various expressions which relate the thermodynamic properties of the dual spacetimes (with one notable exception being the entropy) [22].

Given the above consideration, we can appropriately identify the thermodynamic properties of the CFT as follows:

$$E \equiv E_{CFT} = -\frac{LM}{a}, \quad (18)$$

$$\begin{aligned}
T \equiv T_{CFT} &= \frac{L}{a} T_{dS} \\
&= \frac{1}{4\pi a} \left[\frac{(n+1)a_H}{L} - \frac{(n-1)L}{a_H} + \frac{nL\omega_{n+1}^2 Q^2}{8a_H^{2n-1}} \right], \quad (19)
\end{aligned}$$

$$\begin{aligned}
S \equiv S_{CFT} &= S_{dS} \\
&= \frac{a_H^n V_n}{4G_{n+2}}, \quad (20)
\end{aligned}$$

$$\begin{aligned}
\Phi \equiv \Phi_{CFT} &= -\frac{L}{a} \Phi_{dS} \\
&= -\frac{nL}{4(n-1)} \frac{\omega_{n+1} Q}{aa_H^{n-1}}. \quad (21)
\end{aligned}$$

A commentary regarding the negative sign in Eq.(18) (as well as Eq.(21)) is in order. It has recently been demonstrated that a dS black hole represents an excitation of negative energy [64]. This counter-intuitive result can be attributed to the unusual binding interaction that arises between a positive-mass object and a dS gravitational field [65]. With this observation in mind, we have followed prior works [67, 69] and identified the gravitational energy with the negative of the mass observable.⁸

In similar fashion to the energy, we have reversed the sign of the CFT chemical potential (21). This choice can be further justified by noting that a rotation to the Euclidean sector typically requires an accompanying complexification of charge [72]. Hence, we are inclined to transform $Q^2 \rightarrow -Q^2$, which is effectively the same as introducing a negative sign in Eq.(21).

Let us now suitably define an energy density ($\rho \equiv E/V$), pressure ($p \equiv \rho/n$)⁹, electrostatic potential ($\phi \equiv L\phi_{dS}/a$) and charge density ($\rho_Q \equiv Q/V$); where $V = a^n V_n$. With these definitions, the first and second Friedmann equations (15,16) can now be written as follows:

$$H^2 = \frac{16\pi G}{n(n-1)} \left[\rho - \frac{1}{2}\phi\rho_Q \right] + \frac{1}{a^2}, \quad (22)$$

⁸Note that, in Eq.(18), we have omitted the energy associated with the pure (i.e., $M=0$) dS background. This is consistent with the convention initiated in Refs.[6, 22].

⁹Note that $p = \rho/n$ is the standard equation of state for radiative matter.

$$\dot{H} = -\frac{8\pi G}{(n-1)} [\rho + p - \phi\rho_Q] - \frac{1}{a^2}, \quad (23)$$

where $G = (n-1)G_{n+2}/L$ is the effective Newton constant on the brane.¹⁰ Notably, the cosmological evolution has directly been related to the energy density and pressure of radiative matter, along with an electrostatic (potential) energy density.

It is interesting (and useful later on) to note that the Friedmann equations can also be expressed in following suggestive forms:

$$S_H = \frac{2\pi a}{n} \sqrt{-E_{BH} \left[2(E - \frac{1}{2}\phi Q) - E_{BH} \right]}, \quad (24)$$

$$E_{BH} = n [E + pV - \phi Q - T_H S_H], \quad (25)$$

where we have defined:

$$S_H \equiv (n-1) \frac{HV}{4G}, \quad (26)$$

$$E_{BH} \equiv -n(n-1) \frac{V}{8\pi G a^2}, \quad (27)$$

$$T_H \equiv -\frac{\dot{H}}{2\pi H}. \quad (28)$$

The first Friedmann equation (Eq.(22) or (24)) can further be expressed in the following intriguing manner:

$$S_H^2 = -2(S_B - S_Q)S_{BH} + S_{BH}^2, \quad (29)$$

where:

$$S_B \equiv -\frac{2\pi a}{n} E, \quad (30)$$

$$S_{BH} \equiv \frac{(n-1)}{4Ga} V, \quad (31)$$

$$S_Q \equiv -\frac{2\pi a}{n} \cdot \frac{1}{2} \phi Q. \quad (32)$$

¹⁰This relation between bulk and brane gravitational constants is the usual one for a Randall-Sundrum brane world (generalized to arbitrary dimensionality) [23].

The parameters of Eqs.(26-28,30-31) are identically defined (up to a sign when appropriate) to those found in Ref.[6].¹¹ For an AdS bulk, each of these parameters significantly plays a role with regard to holographic bounds. (See Refs.[6, 22] for a complete discussion.) Here, we have introduced the parameters for illustrative convenience and point out that their respective roles do not necessarily persist in a dS holographic theory. We consider this further in Section 5.

It should be kept in mind, that, assuming an expanding brane universe (i.e., $H > 0$), S_H , S_B , S_{BH} and S_Q are all strictly non-negative. Hence, given the unorthodox form of Eq.(29), one might anticipate that severe cosmological constraints must be imposed on the brane dynamics for a dS bulk spacetime. Again, we defer this discussion until Section 5.

4 Thermodynamics at the Horizon and the Cardy-Verlinde Entropy

As was remarkably demonstrated in Ref.[22] for a bulk AdS spacetime, the CFT thermodynamic relations coincide with the Friedmann equations when the brane crosses the black hole horizon. We will now endeavor to see if the same behavior occurs at the cosmological horizon of a bulk RNdS spacetime.

First, let us compare Eq.(15) for H^2 with the equation that defines the cosmological horizon (5). A brief inspection reveals that the Hubble constant must obey:

$$H = \pm \frac{1}{L} \quad \text{at} \quad a = a_H. \quad (33)$$

The $+$ sign indicates an expanding brane universe, while the $-$ sign describes a brane universe that is contracting. For the sake of simplicity, we will focus on the expanding ($H = +L^{-1}$) case.

Next, let us reconsider the CFT entropy (20). Evidently (and as expected via the second law of thermodynamics), this total entropy remains constant as time varies. However, this is not the case for the entropy density:

$$s \equiv \frac{S}{V} = \frac{(n-1)a_H^n}{4GLa^n}, \quad (34)$$

¹¹We also point out that S_Q of Eq.(32) is identically defined (up to the sign) with the parameter found in Ref.[41].

which clearly evolves along with the brane radius. When the brane crosses the horizon, this entropy density takes on the form:

$$s = \frac{(n-1)H}{4G} \quad \text{at } a = a_H, \quad (35)$$

from which it follows that (cf. Eq.(26)):

$$S = S_H \quad \text{at } a = a_H. \quad (36)$$

It is also instructive to consider the CFT temperature (19) at the horizon-brane coincidence point. Using Eq.(23) for \dot{H} , as well as Eqs.(5,28,33), we obtain the following:

$$T = -\frac{\dot{H}}{2\pi H} = T_H \quad \text{at } a = a_H. \quad (37)$$

Hence, when the brane crosses the horizon, the CFT entropy and temperature take on forms that are simply expressed in terms of the Hubble parameter (and its derivative).

Let us now introduce a quantity that can be readily identified with the Casimir energy [6, 22]:

$$E_C \equiv n[E + pV - \Phi Q - TS]. \quad (38)$$

Given that $T = T_H$, $S = S_H$ and $\Phi = \phi$ at the coincidence point, it follows that (cf. Eq.(25)):

$$E_C = n[E + pV - \phi Q - T_H S_H] = E_{BH} \quad \text{at } a = a_H. \quad (39)$$

We elaborate on the significance of this Casimir energy below.

Let us now return to the case of a general brane radius. It can readily be verified that the CFT thermodynamic properties satisfy the usual first law of thermodynamics. That is:

$$TdS = dE - \Phi dQ + PdV. \quad (40)$$

Much can be revealed if we re-express the first law in terms of densities. In this case:

$$Tds = d\rho - \Phi d\rho_Q + n[\rho + p - \Phi\rho_Q - Ts] \frac{da}{a}, \quad (41)$$

where we have applied the equation of state ($p = \rho/n$) and $dV = nVda/a$ (since $V \sim a^n$).

The combination in the square brackets effectively measures the sub-extensive contribution to the thermodynamic system. With this observation, one can see that the Casimir energy (38) has indeed been appropriately defined. Next, we will derive a more explicit expression for this Casimir contribution.

As a first step in this derivation, it is useful to express the CFT energy density (by way of Eqs.(5,18)) as follows:

$$\begin{aligned}\rho &= -\frac{ML}{a^{n+1}V_n} \\ &= \frac{na_H^n}{16\pi G_{n+2}a^{n+1}} \left[\frac{a_H}{L} - \frac{L}{a_H} - \frac{nL\omega_{n+1}^2 Q^2}{8(n-1)a_H^{2n-1}} \right].\end{aligned}\quad (42)$$

Next, we incorporate $p = \rho/n$, Eq.(34) for s , Eq.(19) for T and Eq.(21) for Φ into the above. This procedure ultimately yields:

$$n[\rho + p - \Phi\rho_Q - Ts] = -\frac{2\gamma}{a^2},\quad (43)$$

where:

$$\gamma \equiv \frac{n(n-1)a_H^{n-1}}{16\pi Ga^{n-1}}.\quad (44)$$

Recalling the definition of the Casimir energy (38), we have:

$$E_C = -\frac{2V\gamma}{a^2} = -\frac{n(n-1)V_na_H^{n-1}}{8\pi Ga}.\quad (45)$$

Notably, this expression has no explicit dependence on the charge Q .

With the above formalism, the entropy density (34) can be directly related to the ‘‘Casimir quantity’’ (i.e., γ). After some tedious manipulations, we find:

$$s^2 = \left(\frac{4\pi}{n}\right)^2 \gamma \left[\rho - \frac{1}{2}\Phi\rho_Q + \frac{\gamma}{a^2} \right].\quad (46)$$

Significantly, this entropy formula has a Cardy-like form [19], with γ playing the role of a ‘‘central charge’’.¹²

¹²In Cardy’s formalism [19], the central charge describes the multiplicity of massless particle species. It is clear that such a quantity should be directly related to the Casimir energy density, as we have found.

Let us now reconsider the special moment when the brane crosses the cosmological horizon. At this coincidence point, the first Friedmann equation (22) can be shown to follow directly from Eq.(46). Furthermore, the second Friedmann equation (23) can be obtained when $a = a_H$ is imposed on Eq.(43). Hence, we have extended the key results of Ref.[22] for the case of a bulk RNdS spacetime.

5 Cosmological Considerations

It is of interest to consider how the holographic bounds of Ref.[6] would have to be revised under this RNdS scenario. First, it is useful if our generalized Cardy-Verlinde formula (46) is re-expressed in the following form:

$$S = \sqrt{\frac{2\pi a}{n} S_C \left[2 \left(E - \frac{1}{2} \Phi Q \right) - E_C \right]}, \quad (47)$$

where we have suitably defined the following Casimir entropy (in analogy with Ref.[6]):

$$\begin{aligned} S_C &\equiv -\frac{2\pi a}{n} E_C \\ &= \frac{(n-1)V_n a_H^{n-1}}{4G}. \end{aligned} \quad (48)$$

Note that S_C is strictly non-negative and depends implicitly (but not explicitly) on M and Q by virtue of its dependence on a_H (cf. Eq.(5)).

With regard to the Casimir entropy, it is particularly significant that:

$$S_C = S_{BH} \quad \text{at} \quad a = a_H, \quad (49)$$

where S_{BH} is the Bekenstein-Hawking entropy of Eq.(31). Recall that the same coincidental behavior was found between the Casimir energy and the “Bekenstein-Hawking energy” (E_{BH}); cf. Eq.(39).

It is not difficult to show that, when $a = a_H$, the above relation (47) for S coincides with Eq.(24) for the “Hubble entropy” (S_H). To illustrate this, let us first consider the following equivalent form of Eq.(47):

$$S^2 = -2(S_B - S_Q) S_C + S_C^2. \quad (50)$$

Here, S_B is the “Bekenstein entropy” of Eq.(30) and we have further defined (in analogy to Eq.(32)):

$$\mathbf{S}_Q \equiv -\frac{2\pi a}{n} \cdot \frac{1}{2}\Phi Q. \quad (51)$$

Comparing Eq.(50) for S with Eq.(29) for S_H , we clearly observe the equivalence of these two entropies when the brane crosses the horizon.

Given the outcomes of the seminal studies [6, 22], one might wonder if such entropic coincidences (at $a = a_H$) actually represent the saturation points of holographic bounds. Before considering this, we point out a significant difference between dS-bulk scenarios and their AdS analogues. As it so happens, the quantity $H^2 a^2$ must always be less than unity in the dS-bulk case. One can see this in a number of ways; for instance, an inspection of Eq.(22), keeping in mind that the CFT energy density is always negative in dS spacetimes.¹³ Hence, in accordance with Verlinde’s classification scheme [6], a dS bulk can only induce a **weakly** self-gravitating brane universe.

It is interesting to note that Verlinde’s “litmus test” for a weakly self-gravitating system, $H^2 a^2 \leq 1$ when $S_B \leq S_{BH}$ [6], translates to our RNdS model as:

$$S_B - S_Q \leq \frac{1}{2}S_{BH} \quad \text{always.} \quad (52)$$

This constraint can readily be obtained from Eq.(29), keeping in mind that S_H^2 , S_B , S_{BH} are strictly non-negative (and we have implicitly assumed that $S_B > S_Q$ ¹³).

In Ref.[6], Verlinde conjectured a universal bound that is based on a holographic upper limit on the degrees of freedom of the CFT as measured by the Casimir energy. That is, $|E_C| \leq |E_{BH}|$ ¹⁴ or equivalently:

$$S_C \leq S_{BH}. \quad (53)$$

We will presume that this intuitive bound continues to hold for the RNdS-bulk scenario without modification. The justification being: (i) the equivalence of these entropies when the brane crosses the horizon and (ii) the

¹³We are assuming here that $|\rho| > \frac{1}{2}|\phi\rho_Q|$. This is equivalent to saying that the radiative energy dominates over the electrostatic energy, which seems a reasonable viewpoint.

¹⁴We have included absolute-value bars to account for the negative brane energy that is induced by the RNdS bulk.

Casimir energy and entropy have no explicit dependence on the charge [41]. However, even with this conjecture, $S \leq S_H$ does **not** necessarily hold up, except for the case of vanishing charge; cf. Eqs.(29,50). (Note that $\mathbf{S}_Q > S_Q$ when $a_H < a$.)

We can still obtain a universal bound for the “total” entropy by way of the following argument. Again assuming that the radiative matter dominates over the electrostatic contribution so that $S_B > \mathbf{S}_Q$, we have from Eqs.(50,53):

$$S < S_C \leq S_{BH}. \quad (54)$$

Not only does a dS black hole excite negative energy, it effectively lowers the entropy from that of a pure (i.e., $M = 0$) dS state. This seems counter-intuitive, but we note that a similar result has been obtained in Refs.[67, 69].

6 Conclusion

In the preceding paper, we have considered a brane universe in a Reissner-Nordstrom-de Sitter background spacetime. The analysis began with the identification of thermodynamics of the RNdS cosmological horizon. We then considered brane dynamics and demonstrated that (with a suitable choice of brane tension) the evolution equations take on a Friedmann-like form.

Next, we applied the dS/CFT correspondence to obtain the thermodynamic properties of a Euclidean CFT that lives on the brane. After which, it was explicitly shown that the CFT thermodynamics coincides with the (generalized) Friedmann equations at the point when the brane crosses the cosmological horizon. Moreover, we were able to derive an expression for the entropy that is readily identifiable as a generalized Cardy-Verlinde formula [19, 6]. In this context, the Casimir energy (i.e., the sub-extensive energy contribution) adopts the role of the Cardy central charge.

Finally, we have considered some of the cosmological implications of our analysis. We found that the “Casimir entropy” coincides with the Bekenstein-Hawking entropy when the brane crosses the horizon. On the basis of this result, we conjectured an upper bound for the Casimir entropy. That is to say, the equivalence of entropies (when the brane and horizon coincide) is really just a saturation of this conjectured upper bound. We again point out that such a bound was previously suggested by Verlinde [6] as a universal consequence of the holographic principle [1, 2].

Although our results came out clearly in support of the proposed dS/CFT correspondence, we have encountered some troubling issues along the way. These include: (i) the total entropy of the CFT being bounded (from above) by the Casimir contribution and (ii) the inaccessibility of dS-induced brane universes to a strongly self-gravitating regime. Such “exotic” behavior (in comparison with analogous AdS scenarios) can seemingly be attributed to the negative energy density that arises on the brane. This unusual brane property can, in turn, be linked (via the dS/CFT duality) to dS black holes describing an excitation of negative energy [64, 65]. This strange effect should probably be better understood before the dS/CFT correspondence can be put on equal footing with its AdS analogue. Furthermore, there remains the open issue of how to incorporate the thermodynamics of the dS black hole horizon into the proposed duality.

Suffice it to say, the dS/CFT correspondence requires further investigation; although considerable progress has definitely been made.

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